Topological Degeneracy of Quantum Hall Fluids

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Abstract

We present a simple approach to calculate the degeneracy and the structure of the ground states of non-abelian quantum Hall (QH) liquids on the torus. Our approach can be applied to any QH liquids (abelian or non-abelian) obtained from the parton construction. We explain our approach by studying a series of examples of increasing complexity. When the effective theory of a non-abelian QH liquid is a non-abelian Chern-Simons (CS) theory, our approach reproduces the well known results for the ground state degeneracy of the CS theory. However, our approach also apply to non-abelian QH liquids whose effective theories are not known and which cannot be written as a non-abelian CS theory. We find that the ground states on a torus of all non-abelian QH liquids obtained from the parton construction can be described by points on a lattice inside a "folded unit cell." The folding is generated by reflection, rotations, etc. Thus the ground state structures on the torus described by the "folded unit cells" provide a way to (at least partially) classify non-abelian QH liquids obtained from the parton construction.

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1 Introduction

It has become increasingly clear that the Quantum Hall (QH) liquids[1, 2] are not merely disordered liquids: they are quantum liquids with extremely rich and totally new internal structures (or topological orders)[3]. Thus QH liquids represent a new class of matter. Although we still do not have a complete theory of this new kind of order, we do know that QH liquids can be divided into two classes – abelian[4, 5, 6] and non-abelian[7, 8]. The effective theory of abelian QH liquids is known to be the $U(1)$ Chern-Simons (CS) theory [5, 6], and because of that we have a classification of all abelian QH liquids [11] in terms of the so-called K matrix.

In contrast, the effective theories of many known non-abelian QH liquids are unknown. The problem is not because we know too little about a non-abelian QH liquid so that we cannot deduce the effective theory. In many cases, we know a lot about the low energy properties of a QH liquid, and still do not know its effective theory. This is simply because the correct effective theory has not been named yet. Giving a name is easy, but giving a proper name is hard. Giving a proper name which carries meaningful information amounts to the task of classifying non-abelian QH liquid, and so far we do not know how to do this.

In this paper we concentrate on the physical properties of non-abelian states on the torus, and use a simple approach to calculate the ground state degeneracy of non-abelian states. We assume no prior knowledge of the subject. Our approach is down to earth, and we proceed through specific examples.

First we test our approach on abelian states. Then we study the non-abelian state with wave function $(\chi_q(\{z_i\}))^2$ where χ_q is the wave function with q filled Landau levels. Using the $SU(2)_q$ CS effective theory of the $(\chi_q(z_i))^2$ state[8], we find that the $(\chi_q(z_i))^2$ state has $q+1$ degenerate ground states on a torus. The $q+1$ degeneracy of the $SU(2)_q$ CS theory has been calculated before, using a powerful mathematical approach based on algebraic geometry, topological theory, and Lie algebraic theory [9, 10]. We hope that our discussion will be more accessible to the non-mathematical reader. Next, we study a slightly more complicated non-abelian state with wave

function $\chi_1(\{z_i\})(\chi_q(\{z_i\}))^2$. We are able to set up a simple model which describes the $(q+1)(q+2)/2$ degenerate ground states of the $\chi_1(\{z_i\})(\chi_q(\{z_i\}))^2$ QH liquid on the torus. It is not clear, however, whether our model can be derived from a named topological field theory. Since the model is given by the $U(1)_{2q+4} \times SU(2)_q$ CS theory (which has $2(q+1)(q+2)$ ground states on the torus) with some additional projections, we call our effective theory $(U(1)_{2q+4} \times SU(2)_q)/Z_2$ theory. Then, we study the non-abelian state with wave function $[\chi_q(\{z_i\})]^3$ (associated with $SU(3)$ CS theory) to demonstrate more features of the ground states on torus. In particular we show that, on torus, the non-abelian state $[\chi_q(\{z_i\})]^3$ can described by a $U(1) \times U(1)$ CS theory plus some projections.

Although we only discussed some simple examples, the approach used here can be applied to more complicated non-abelian states. Through those studies we see some general patterns. We hope those patterns will shed light on how to classify topological orders in non-abelian QH liquids. We outline such a classification in the concluding section.

2 Abelian FQH states

It is well known by now that the fractional quantum Hall (FQH) fluid can be represented effectively by a Chern-Simons CS field theory [5, 6, 11]

$$S = \int d^2x\ dt\ \frac{1}{4\pi}\ K_{IJ}\ \epsilon^{\mu\nu\lambda}\ a^I_\mu\ \partial_\nu\ a^J_\lambda \tag{1}$$

with an integer valued matrix K. The theory is topological, with low energy properties characterized by degenerate ground states. In a classic paper [12] on the subject, Wen showed that the ground state degeneracy of (1) when quantized on a manifold of genus g is given by

$$D = (det\ K)^g \tag{2}$$

In this paper, we study the simple case when the manifold is a torus ($g = 1$). First let we assume the matrix K is 1×1 and equal to an integer k. In this case the CS theory (1) is the effective theory of filling fraction $\nu = 1/k$ Laughlin state,

and D is just the ground state degeneracy of the Laughlin state. Let us call the CS theory with $K = k$ the $U(1)_k$ CS theory. The approach used below and in the next section for abelian and non-abelian CS theories are not entirely new (see Ref. [9]). We present it here for the purpose of introducing proper notations and concepts for later discussions. We also present our introduction to the CS theory (abelian or non-abelian) in a way which is easy for people not in the field of field/string theory to understand.

Wen [12] determined the ground state degeneracy D by adding to S the non-topological Maxwell term

$$\int d^2x\ dt\ \sqrt{g}\ g^{\mu\lambda}\ g^{\nu\sigma}\ \frac{1}{f^2}\ F_{\mu\nu}\ F_{\lambda\sigma} \tag{3}$$

with f^2 some coupling constant. Here $g^{\mu\nu}$ denote the metric of the manifold. On a torus, the ground state properties are determined by constant (independent of (x_1, x_2) but of course dependent on time) gauge potentials

$$a_0(x_1, x_2, t) = 0, \quad a_1(x_1, x_2, t) = \frac{2\pi x(t)}{L_1}, \quad a_2(x_1, x_2, t) = \frac{2\pi y(t)}{L_2} \tag{4}$$

where (L_1, L_2) are the size of the torus. The dynamics of the constant gauge potentials is described by the Lagrangian obtained by inserting (4) into (1) and (3) (with m determined by the Maxwell coupling $1/f^2$)

$$L = \pi k(y\dot{x} - x\dot{y}) + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \tag{5}$$

The "large" gauge transformations $V = e^{i2\pi x_1/L_1}$ and $V = e^{i2\pi x_2/L_2}$ transform a constant gauge potential to another constant gauge potential, and induce (under the gauge transformation $a_\mu \to a_\mu - iV^{-1}\partial_\mu V$) the following changes: $(x, y) \to (x+1, y)$ and $(x, y) \to (x, y+1)$. Since (x, y), $(x+1, y)$, and $(x, y+1)$ are related by gauge transformation, they represent equivalent point

$$(x, y) \sim (x+1, y) \sim (x, y+1) \tag{6}$$

Thus Lagrangian in (5) describes a mass m particle on a torus with a uniform "magnetic field." The total number of flux quanta is k, which leads to a k-fold degenerate ground state.

The defining characteristic of a topological theory such as (1) is that it does not depend on the metric. We should be able to determine the ground state of a topological field theory without having to add a regulating term which breaks the topological character of the theory. In the following, we will determine the ground state degeneracy without adding the regulating term.

Given (taking $m = 0$ in (5))

$$S = \int dt \; L = \int dt \; 2\pi k \; x \; \dot{y} \tag{7}$$

we have

$$\frac{\delta L}{\delta \dot{y}} = 2\pi \; kx \tag{8}$$

and so

$$[x, y] = \frac{i}{2\pi k} \tag{9}$$

If we regard y as the position variable, the conjugate momentum is given by $p = 2\pi kx$. The Hamiltonian H vanishes,

$$H = p \; \dot{y} - L = 0 \tag{10}$$

a hallmark of a topological theory. Thus the Schrödinger equation just reads

$$0 \cdot \psi = E\psi \tag{11}$$

How then do we determine the wave functions $\psi(y)$?

Naively, any function $\psi(y)$ will have zero energy and would qualify as a ground state wave function. But such a wave function would in general not satisfy the equivalence condition (6)! The allowed wave functions are determined by the requirement that the particle lives on a torus with coordinates (x, y) such that $x \sim x + 1$ and $y \sim y + 1$. In other words, y and $y + 1$ really represent the same point and hence we must require $\psi(y) = \psi(y + 1)$. This periodicity condition implies that

$$\psi(y) = \sum_{n = -\infty}^{\infty} c_n \; e^{i2\pi ny} \tag{12}$$

with n an integer. To impose the periodicity condition in the x direction $x \sim x + 1$, we need to make a Fourier transformation to obtain the wave function in the "momentum" space. Canonical conjugation gives $p = i\frac{\partial}{\partial y}$ and so

$$\tilde{\psi}(p) = \sum c_n \, \delta(p - 2\pi \, n) \tag{13}$$

It is useful (if only to get rid of the (2π)'s, but more importantly to emphasize the equal status of x and y) to recall $p = 2\pi \, k \, x$ and hence to define a wave function in the x coordinate

$$\phi(x) = \sum c_n \, \delta(kx - n) \tag{14}$$

The condition $x \sim x + 1$ now implies that

$$c_n = c_{n+k} \tag{15}$$

We have thus reached the conclusion that the ground state degeneracy D is k, since there are k independent c_n's, namely $c_1, c_2, \ldots, c_k$. We have thus shown how to determine the ground state degeneracy without breaking the topological character of the theory.

It is useful to define the periodic delta function

$$\delta^P(y) \equiv \sum_{l=-\infty}^{\infty} e^{i2\pi l y} \tag{16}$$

equal to 1 (up to some irrelevant overall infinite constant) if y is an integer, and 0 otherwise. It is also convenient to write, for any integer n,

$$n = lk + m \tag{17}$$

(with l an integer) and define $[n]_k \equiv m$ as the reduced part of n. We will suppress the index k on $[n]_k$ if there is no ambiguity. With these definitions we can re-write (12) as

$$\begin{aligned} \psi(y) &= \left(\sum_{m=1}^{k} c_m \, e^{i2\pi m y} \right) \left(\sum_{l=-\infty}^{\infty} e^{i2\pi l k y} \right) \\ &= g(y) \, \delta^P(ky) \end{aligned} \tag{18}$$

Again writing $n = lk + m$, we obtain from (14) that $[kx] = m$. Since $c_n = c_m$ by periodicity, we have

$$\phi(x) = c_{[kx]}\, \delta^P(kx) \tag{19}$$

This shows that $\phi(x)$ and $\psi(y)$ are indeed on the same footing, and that $\psi(y)$ is, in the sense described here, just the Fourier transform of c_m. We note that $\phi(x)$ and $\psi(y)$ are proportional to $\delta^P(kx)$ and $\delta^P(ky)$ respectively. Thus the position coordinates x and y are both quantized to be $\frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1$. Of course, x and y cannot be simultaneously diagonalized. More precisely, the wave function $\psi(x)$ is non-zero only when x take on certain discrete values, (and similarly with x replaced by y.)

We mention here for later use that it is now of course straightforward to generalize the case given in (1). The effective point particle quantum mechanics is described by the Lagrangian

$$L = 2\pi\, K_{IJ}\, x_I\, \dot{y}_J \tag{20}$$

(repeated indices in I, J summed) and the commutation relation

$$[x_I,\ y_J] = \frac{i}{2\pi}\, (K^{-1})_{IJ} \tag{21}$$

The wave function in the y coordinates is given by

$$\psi(\mathbf{y}) = \sum_{\mathbf{n}} c_{\mathbf{n}}\, e^{i2\pi\ \mathbf{n}\cdot\mathbf{y}} \tag{22}$$

where n_I, the components of $\mathbf{n}$, are integers. The corresponding wave function in the x-coordinates is

$$\psi(\mathbf{x}) = \sum_{\mathbf{n}} c_{\mathbf{n}}\, \delta\left(x_I - (K^{-1})_{IJ}\, n_J\right) \tag{23}$$

3 $SU(2)$ non-abelian FQH states

Next, let us calculate the ground states degeneracy of a simple class of non-abelian QH states. It was pointed out [8] that the QH liquid described by wave function $(\chi_q(z_1, ..., z_N))^2$ (where χ_q is the fermion wave function with q filled Landau levels)

is a non-abelian QH state, whose effective theory is the $SU(2)_q$ CS theory (*i.e.* the $SU(2)$ level-q CS theory). Let us first recall how the non-abelian states [7] can be constructed, using the rather physical parton construction [8]. For the sake of pedagogical clarity, we focus on a specific (but unphysical) example.

We imagine that at short distances the electron can be cut into two constituents ("partons"), each of charge $e_0 = e/2$. The long distance physics of the resulting Hall fluid should be independent of the details of the short distance dynamics. This can be explained very simply in terms of wave functions. Denote the coordinates of the partons by $\{z_i^\alpha\}$, $\alpha = 1,2$ and $i = 1,2,\ldots,N$, with N electrons in the system. Let each species of partons fill q Landau levels, and denote the corresponding wave function by χ_q. The wave function of the entire fluid is then given by

$$\Psi \sim \chi_q \left(z_1^{(1)},\ldots,z_N^{(1)}\right) \chi_q \left(z_1^{(2)},\ldots,z_N^{(2)}\right) \tag{24}$$

We now have to tie the "partons" together to form the electrons. This is done by setting $z_j^{(1)} = z_j^{(2)} = z_j$ in the wave function Ψ. For instance, for $q = 1$, we have $\chi_1 = \prod_{i>j}(z_i - z_j)$, and so we obtain $\Psi \sim \prod_{i>j}(z_i - z_j)^2$, which is nothing but the $\nu = \frac{1}{2}$ Laughlin state (for bosonic electrons), as Laughlin [2] taught us.

In field theoretic language, before we bind the "partons" together into electrons, we have the Lagrangian

$$\begin{aligned}\mathcal{L} &= i\psi_1^\dagger\partial_t\psi_1 + \frac{1}{2m}\psi_1^\dagger(\partial_i - ie_0A_i)^2\psi_1 \\ &\quad + i\psi_2^\dagger\partial_t\psi_2 + \frac{1}{2m}\psi_2^\dagger(\partial_i - ie_0A_i)^2\psi_2\end{aligned} \tag{25}$$

with ψ_1, ψ_2 corresponding to the two parton fields. We glue the two "partons" together by coupling them to an $SU(2)$ gauge potentials a_μ:

$$\mathcal{L} = i\psi^\dagger(\partial_t - ia_0)\psi + \frac{1}{2m}\psi^\dagger(\partial_i - ie_0A_i - ia_i)^2\psi \tag{26}$$

where $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ and a_μ are hermitian traceless 2 by 2 matrices. Now we can integrate out $\psi_{1,2}$ (see for example Ref. [6]) and obtain the effective theory:

$$\mathcal{L} = \frac{2qe_0^2}{4\pi}\,\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{q}{4\pi}\mathrm{Tr}\epsilon^{\mu\nu\lambda}(a_\mu\partial_\nu a_\lambda + \frac{i}{3}a_\mu a_\nu a_\lambda) \tag{27}$$

All we need from Ref. [6] is the result that given

$$\mathcal{L} \;=\; i\psi^\dagger \partial_t \psi \;+\; \frac{1}{2m}\psi^\dagger(\partial_i - ie_0 A_i)^2\psi \tag{28}$$

and with ψ filling q Landau levels we obtain the effective Lagrangian

$$\mathcal{L} \;=\; \frac{qe_0^2}{4\pi}\,\epsilon^{\mu\nu\lambda}\; A_\mu \partial_\nu A_\lambda \tag{29}$$

upon integrating out ψ. There is also an analogous formula for the non-abelian sector.

The first term tells us that the this state has a filling fraction $\nu \;=\; 2qe_0^2/e^2 \;=\; q/2$. The second term describes a level q $SU(2)$ (denoted by $SU(2)_q$) Chern-Simons CS effective theory, which determines the ground state properties of the QH liquid on compact spaces.

Starting from the $SU(2)_q$ CS theory (27), we choose the $a_0 = 0$ gauge to calculate the ground states on a torus. In this gauge we need to enforce the constraint of zero field strength: $f_{ij} = 0$. Introduce Wilson loop operators $U_c \equiv P[e^{i\oint_c dx_\mu a_\mu}] \in SU(2)$ (where $P[...]$ is a path ordered product). For a contractable path c, we have trivially $U_c = 1$ due to the constraint. On a torus, all the gauge invariant quantities are contained in the two Wilson loop operators for the two non-contractable loops $c_{1,2}$ in the x_1 and x_2 directions: $U_1 = P[e^{i\oint_{c_1} dx_\mu a_\mu}]$ and $U_2 = P[e^{i\oint_{c_2} dx_\mu a_\mu}]$. Since $U_1U_2U_1^\dagger U_2^\dagger$ is a Wilson loop operator for a contractable loop, we have $U_1U_2U_1^\dagger U_2^\dagger = 1$, and U_1 commutes with U_2. Making a global $SU(2)$ gauge transformation, we can make $U_{1,2}$ have the form

$$U_1 = e^{i2\pi x\tau_3}, \qquad U_2 = e^{i2\pi y\tau_3} \tag{30}$$

This corresponds to spatially constant gauge potentials: writing $a_i = a_i^l \tau_l$ with τ_l the usual Pauli matrices we have

$$a_i^{1,2} = 0, \quad a_1^3(x_1, x_2, t) = 2\pi x(t)/L_1, \quad a_2^3(x_1, x_2, t) = 2\pi y(t)/L_2 \tag{31}$$

We see that the $SU(2)$ CS theory has at low energies non-trivial physical degrees of freedom described by (x, y) (just as in (5)). Classically different values of (x, y) correspond to different degenerate physical states, and there are infinite number of

degenerate ground states. However, in quantum theory, x and y do not commute and the uncertainty relation leads to only a finite number of degenerate ground states. To obtain the commutator between x and y, we insert (31) into (27) to obtain the effective theory

$$S = \int dt\ 2\pi q(x\dot{y} - y\dot{x}). \tag{32}$$

This is identical to the effective theory for the $U(1)_k$ theory (7) with the identification $k = 2q$. The resulting commutator is

$$[x, y] = \frac{i}{2\pi k} \tag{33}$$

Again, under a large gauge transformation, we have

$$x \sim x + 1, \quad y \sim y + 1 \tag{34}$$

Thus, naively one might think that the $SU(2)_q$ theory is described by (32) and (34) which is nothing but a $U(1)_k$ theory with $k = 2q$. But that would be wrong. In fact, the $SU(2)_q$ theory is not equivalent to the $U(1)_{2q}$ theory, because the $SU(2)$ CS theory contains an additional global $SU(2)$ gauge transformation that changes a^3 to $-a^3$. This gauge transformation imposes an additional equivalence condition

$$(x, y) \sim (-x, -y) \tag{35}$$

Eqs. (32), (34), and (35) form a complete description of the $SU(2)_q$ theory on the torus.

From the above discussion, we see that the $SU(2)_q$ states can be obtained from the $U(1)_{k=2q}$ states. The k states in the $U(1)_k$ theory is given by

$$\psi(y) \ = \ \sum c_n\ e^{i2\pi ny} \tag{36}$$

or

$$\phi(x) \ = \ \sum c_n\ \delta(kx \ - \ n) \tag{37}$$

with $c_n = c_{n+k}$. The additional condition (35) implies that only those states that satisfy $\psi(y) = \psi(-y)$ and $\phi(x) = \phi(-x)$ can belong to the $SU(2)_q$ theory. This requires $c_n = c_{-n}$. Thus the $SU(2)_q$ CS theory, as well as the non-abelian FQH state described by $(\chi_q)^2$, has $q+1$ degenerate ground states, corresponding to the $q+1$ independent coefficients $c_0, c_1, \dots, c_q$.

It turns out that $SU(2)_1$ CS theory represents a special case. We note that when $q = 1$, the requirement $c_n = c_{-n}$ does not remove any states. This agrees with the known result that the $SU(2)_1$ CS theory is equivalent to the $U(1)_2$ CS theory.

4 $U(1) \times SU(2)$ non-abelian FQH states

In this section we are going to discuss a non-abelian state which is closely related to the one discussed above, but which is physical. The electron is a fermion and we would like to split it into three fermionic partons. Thus we first split an electron into three different partons of electric charge $e_0 = q/(q+2)$, $e_1 = e_2 = 1/(q+2)$ (so that $e_0 + e_1 + e_2 = 1$), and write the above wave function as $\chi_1(z_1^{(0)}, .., z_N^{(0)})\chi_q(z_1^{(1)}, ..., z_N^{(1)})\chi_q(z_1^{(2)}, ..., z_N^{(2)})$. This non-abelian state has a wave function $\chi_1(z_1, .., z_N)(\chi_q(z_1, ..., z_N))^2$. The effective theory for the partons is given by

$$\begin{aligned}\mathcal{L} &= i\psi_0^\dagger\partial_t\psi_0 + \frac{1}{2m}\psi_0^\dagger(\partial_i - ie_0A_i)^2\psi_0 \\ &\quad + i\psi_1^\dagger\partial_t\psi_1 + \frac{1}{2m}\psi_1^\dagger(\partial_i - ie_1A_i)^2\psi_1 \\ &\quad + i\psi_2^\dagger\partial_t\psi_1 + \frac{1}{2m}\psi_2^\dagger(\partial_i - ie_2A_i)^2\psi_2 \end{aligned} \tag{38}$$

The above effective theory describes three independent QH fluids of filling fraction $\nu_0 = 1$, $\nu_1 = \nu_2 = q$. Now we include a $U(1)$ and an $SU(2)$ gauge field, b_μ and a_μ, to recombine partons together to form an electron:

$$\begin{aligned}\mathcal{L} &= i\psi_0^\dagger(\partial_t - 2ib_0)\psi_0 + \frac{1}{2m}\psi_0^\dagger(\partial_i - ie_0A_i - 2ib_i)^2\psi_0 \\ &\quad + i\psi^\dagger(\partial_t - ia_0 + ib_0)\psi + \frac{1}{2m}\psi^\dagger(\partial_i - ie_1A_i - ia_i + ib_i)^2\psi \end{aligned} \tag{39}$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and a_μ are 2 by 2 matrices. Now we can integrate out $\psi_{0,1,2}$ (see (29) or Ref. [6]) and obtain the effective theory for the $\chi_1\chi_q^2$ state:

$$
\begin{aligned}
\mathcal{L} \;=\;& \frac{e_0^2+2qe_1^2}{4\pi}A_\mu\partial_\nu A_\lambda\epsilon^{\mu\nu\lambda} + \frac{2^2+2q}{4\pi}b_\mu\partial_\nu b_\lambda\epsilon^{\mu\nu\lambda} + \frac{q}{4\pi}\mathrm{Tr}\epsilon^{\mu\nu\lambda}(a_\mu\partial_\nu a_\lambda + \frac{i}{3}a_\mu a_\nu a_\lambda) \\
& -\frac{4e_0+4qe_1}{4\pi}A_\mu\partial_\nu b_\lambda\epsilon^{\mu\nu\lambda}
\end{aligned} \tag{40}
$$

The first term tells us that the $\chi_1\chi_q^2$ state has a filling fraction $\nu = e_0^2 + qe_1^2 + qe_1^2 = q/(q+2)$. The next two terms describe a $U(1)_{2q+4}\times SU(2)_q$ CS effective theory, which determines the ground state properties of the QH liquid on compact spaces.

According to the results we have thus far, the $U(1)_{2q+4}\times SU(2)_q$ CS theory has $(2q+4)(q+1)$ degenerate ground states on the torus. Thus one may naively expect that the $\chi_1\chi_q^2$ state also has $(2q+4)(q+1)$ degenerate ground states on the torus. However this results cannot be right, since when $q=1$ the $\chi_1\chi_q^2$ state is nothing but the $\nu=1/3$ Laughlin state and should have 3 degenerate ground states instead of 12 as implied by $(2q+4)(q+1)$. Therefore despite the above "derivation", the $U(1)_{2q+4}\times SU(2)_q$ CS theory cannot be the correct effective theory for the $\chi_1\chi_q^2$ state. As we will see later, however, the correct effective theory can be obtained from the $U(1)_{2q+4}\times SU(2)_q$ CS theory.

Recall that the $U(1)_{2q+4}\times SU(2)_q$ CS theory on a torus can be described by four degrees of freedoms (x,y) and (x',y'). The corresponding gauge fields are given by

$$
b_1(x_1,y_1,t) = 2\pi\frac{x(t)}{L_1}, \qquad b_2(x_1,y_1,t) = 2\pi\frac{y(t)}{L_2} \tag{41}
$$

and

$$
a_1(x_1,y_1,t) = 2\pi\frac{x'(t)}{L_1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad a_2(x_1,y_1,t) = 2\pi\frac{y'(t)}{L_2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{42}
$$

This reduces the effective theory (40) to

$$
L \;=\; \pi(2q+4)(x\dot{y} - y\dot{x}) + 2\pi q(x'\dot{y}' - y'\dot{x}'). \tag{43}
$$

As noted before, large gauge transformations give us some equivalence conditions. For example the $U(1)$ large gauge transformation $\exp\left(i\frac{2\pi x_1}{L_1}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}\right)$ that

acts on $\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}$ shifts x to $x+1$. This kind of large gauge transformations leads to the equivalence condition

$$x \sim x+1, \quad y \sim y+1 \tag{44}$$

Similarly, the $SU(2)$ large gauge transformations (such as $\exp\left(i\frac{2\pi x_1}{L_1}\begin{pmatrix}0 & & \\ & 1 & \\ & & -1\end{pmatrix}\right)$) leads to

$$x' \sim x'+1, \quad y' \sim y'+1 \tag{45}$$

The $SU(2)$ CS theory also has an additional reflection equivalence condition

$$(x', y') \sim (-x', -y') \tag{46}$$

Equations (43), (44), (45), and (46) describe the $U(1)_{2q+4} \times SU(2)_q$ CS theory on the torus and has $(2q+4)(q+1)$ degenerate ground states.

However, for our theory we have an additional large gauge transformation which mixes the $U(1)$ and the center of $SU(2)$. The large gauge transformations are given by $\exp\left(i\frac{2\pi x_1}{L_1}\begin{pmatrix}1 & & \\ & -1 & \\ & & 0\end{pmatrix}\right)$ and $\exp\left(i\frac{2\pi x_2}{L_2}\begin{pmatrix}1 & & \\ & -1 & \\ & & 0\end{pmatrix}\right)$ and gives rise to the following equivalence conditions

$$(x, x') \sim (x+\frac{1}{2}, x'-\frac{1}{2}), \quad (y, y') \sim (y+\frac{1}{2}, y'-\frac{1}{2}) \tag{47}$$

since $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

We believe that equations (43), (44), (45), (46), and (47) describe the the correct effective theory for the $\chi_1\chi_q^2$ state on torus. Because the correct effective theory is obtained from the $U(1)_{2q+4} \times SU(2)_q$ CS theory by applying the additional equivalence condition (47), we will call it the $(U(1)_{2q+4} \times SU(2)_q)/Z_2$ theory. The edge excitations of the $\chi_1\chi_q^2$ state is discussed in Ref. [8], which is described by the $U(1) \times SU(2)_q$ KM algebra matching very well with the bulk effective theory.

To obtain the ground state degeneracy of the $\chi_1\chi_q^2$ state, we start with the $U(1)_{2q+4} \times U(1)_{2q}$ theory. The states are given by

$$\phi(x, x') = c_{[(2q+4)x],[2qx']}\delta^P((2q+4)x)\delta^P(2qx') \tag{48}$$

where the coefficient $c_{n,n'}$ (with n, n' =integer) satisfy

$$c_{n,n'} = c_{n+2q+4,n'} = c_{n,n'+2q} \tag{49}$$

as a consequence of the equivalence conditions (44), and (45). The condition $(x, x') \sim (x + \frac{1}{2}, x' - \frac{1}{2})$ in (47) is satisfied by requiring

$$c_{n,n'} = c_{n+q+2,n'-q} \tag{50}$$

and $(y, y') \sim (y + \frac{1}{2}, y' - \frac{1}{2})$ in (47) is satisfied by requiring

$$c_{n,n'} = 0 \quad \text{if } n + n' = odd \tag{51}$$

This can be seen from the relation

$$\psi(y, y') \;=\; \sum_{n,n'} c_{n,n'} \; e^{i2\pi(ny+n'y')} \tag{52}$$

The reflection condition (46) gives us

$$c_{n,n'} = c_{n,-n'} \tag{53}$$

In Fig. 1, the circles represent $(2q + 4)(2q)/4$ independent $c_{n,n'}$ after imposing (49), (50) and (51). The filled circles represent $\frac{(2q+4)(2q)}{4\times 2} + \frac{2q+4}{2\times 2} = \frac{(q+1)(q+2)}{2}$ independent $c_{n,n'}$ after imposing the additional reflection condition (53). Therefore the $\chi_1\chi_q^2$ state has

$$D \;=\; (q + 1)(q + 2)/2 \tag{54}$$

degenerate ground states on torus.

Note that when $q = 1$ the $\chi_1\chi_q^2$ state is just the $\nu = 1/3$ Laughlin state and $\frac{(q+1)(q+2)}{2} = 3$ is the expected ground state degeneracy. When $q = 2$ the $\nu = 1/2$ $\chi_1\chi_2^2$ state has six degenerate ground states.

By an elegant argument, Tao and Wu [14] threaded a solenoid through the quantum Hall geometry proposed by Laughlin and showed that as the flux of the solenoid

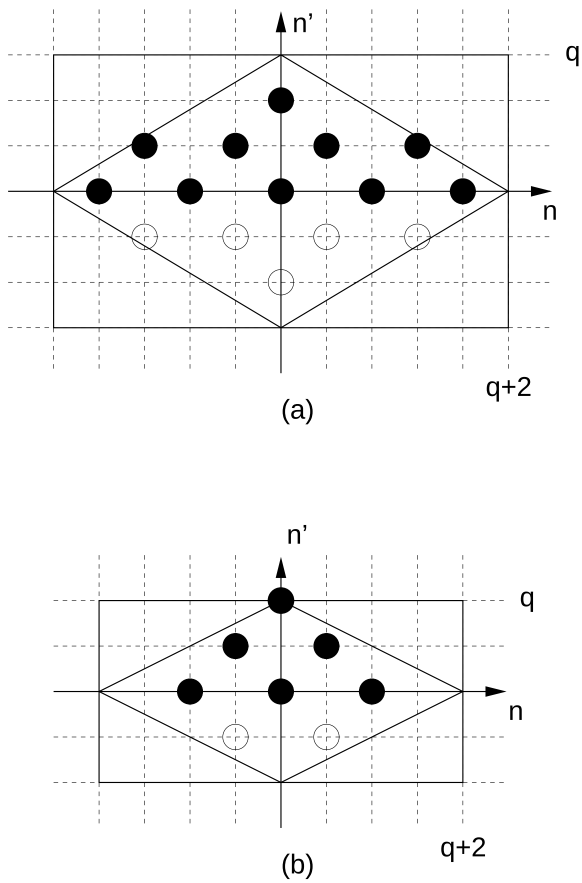

Figure 1: The ground states of $\chi_1 \chi_q^2$ non-abelian state for (a) $q = 3$ and (b) $q = 2$.

increases by quantized units, if the system goes through p degenerate states before returning to its original state, an integer number n of electrons are transported. Thus, the Hall conductance or filling fraction is given by $\nu = n/p$ while the degeneracy $D = mp$ is an integer m multiple of p. That is the ground state degeneracy D is an integer multiple of the denominator of ν. We note that, for our results, the denominator of the filling fraction $\frac{q}{q+2}$ is always a factor of the degeneracy $\frac{(q+1)(q+2)}{2}$. This is consistent with the above result.

5 $SU(3)$ non-abelian FQH states

Now let us calculate the ground states degeneracy of $SU(3)$ non-abelian QH states, whose wave functions are given by $(\chi_q(z_1, ..., z_N))^3$ and which have a filling fraction $q/3$. In the parton construction, we cut the electron into three pieces, each of charge $e_0 = e/3$. We put the partons into Landau levels. Finally, we glue the partons together into electrons.

At the gluing stage, we have a choice. We can do the gluing either with an $SU(3)$ gauge field, thus obtaining a non-abelian CS theory, or with two $U(1)$ gauge fields a_μ and b_μ, thus obtaining an abelian $(U(1))^2$ CS theory. We will discuss the second option first as it is conceptually somewhat simpler.

As before, we have

$$\begin{aligned}
\mathcal{L} \;=\;& i\psi_1^\dagger \left(\partial_t - i(a_0 + b_0)\right)\psi_1 \;+\; \frac{1}{2m}\psi_1^\dagger(\partial_i - ie_0A_i - i(a_i + b_i))^2\psi_1 \\
&+ i\psi_2^\dagger \left(\partial_t - i(-a_0 + b_0)\right)\psi_2 \;+\; \frac{1}{2m}\psi_2^\dagger(\partial_i - ie_0A_i - i(-a_i + b_i))^2\psi_2 \\
&+ i\psi_3^\dagger \left(\partial_t - i(-2b_0)\right)\psi_3 \;+\; \frac{1}{2m}\psi_3^\dagger(\partial_i - ie_0A_i - i(-2b_i))^2\psi_3
\end{aligned} \tag{55}$$

(Note that we need two gauge fields. Suppose we introduce only b_μ. Then we would have the bound states $\psi_1\psi_1\psi_3$, $\psi_2\psi_2\psi_3$, as well as $\psi_1\psi_2\psi_3$. There would be three different kinds of electrons.)

Filling q Landau levels with ψ_1, ψ_2, ψ_3, and integrating out the ψ fields, we obtain

$$\begin{aligned}\mathcal{L} &= \frac{3qe_0^2}{4\pi}\,\epsilon^{\mu\nu\lambda}\; A_\mu \partial_\nu A_\lambda \\ &+ \frac{q}{4\pi}\epsilon^{\mu\nu\lambda}(2a_\mu\partial_\nu a_\lambda \;+\; 6b_\mu\partial_\nu b_\lambda)\end{aligned} \tag{56}$$

Introducing Wilson loops as before, we write

$$a_1 \;=\; \frac{2\pi}{L_1}x(t),\;\; a_2 \;=\; \frac{2\pi}{L_2}y(t) \tag{57}$$

and

$$b_1 \;=\; \frac{2\pi}{L_1}x'(t),\;\; b_2 \;=\; \frac{2\pi}{L_2}y'(t) \tag{58}$$

We insert (57) and (58) into (55) to obtain the effective action for the low energy degrees of freedom:

$$S = \int dt\; 2\pi q(x\dot{y} - y\dot{x}) \;+\; 6\pi q\;(x'\dot{y'} \;-\; y'\dot{x'}) \tag{59}$$

Performing a $U(1)_a$ transformation of the form $U_a \;=\; e^{i2\pi x_1/L_1}$, we conclude that $x \;\sim\; x+1$ with y, x', y' unchanged. Similarly, performing a U_b transformation, we conclude that $x' \;\sim\; x'+1$ with x, y, y' unchanged. Furthermore, we can also perform the corresponding transformations along the x_2 direction and change y and y' respectively. We can thus interpret (59) as describing the motion of two particles on a torus of size (1,1).

However, there is an additional slightly subtle point: we can perform a gauge transformation using the diagonal subgroup of $U(1)_a \;\times\; U(1)_b$. More precisely, we transform $\psi_1 \;\rightarrow\; e^{i2\pi x_1/L_1}\psi_1$ and $\psi_3 \;\rightarrow\; e^{-i2\pi x_1/L_1}\psi_3$, leaving ψ_2 unchanged. This implies

$$S \;:\; (x, x') \;\sim\; \left(x \;+\; \frac{1}{2},\; x' \;+\; \frac{1}{2}\right) \tag{60}$$

with y, y' left invariant. Similarly, performing the corresponding transformation along the x_2 direction we have $(y, y') \;\sim\; \left(y \;+\; \frac{1}{2},\; y' \;+\; \frac{1}{2}\right)$. In other words, various points in the phase space of the two quantum particles have to be identified.

We can now write down the wave function in the y basis as

$$\psi(y, y') \;=\; \sum_{n,n'} c_{n,n'}\; e^{i2\pi(ny \;+\; n'y')} \tag{61}$$

and correspondingly in the x basis as

$$\phi(x, x') \;=\; \sum_{n,n'} c_{n,n'}\; \delta(2qx \;-\; n)\; \delta(6qx' \;-\; n') \tag{62}$$

The torus boundary condition implies that n and n' are integers and

$$c_{nn'} \;=\; c_{n+2q,n'} \;=\; c_{n,n'+6q} \tag{63}$$

Furthermore, the equivalence relation (60) implies that

$$c_{nn'} \;=\; c_{n-q,n'-3q} \tag{64}$$

Next, we note that the Lagrangian (55) enjoys three discrete interchange symmetries $u \;:\; \psi_1 \;\leftrightarrow\; \psi_2$; $v \;:\; \psi_2 \;\leftrightarrow\; \psi_3$; and $w \;:\; \psi_1 \;\leftrightarrow\; \psi_3$; with the corresponding operations on the two gauge fields a and b. Indeed, mathematically, the three operations u, v, and w generate the permutation group S_3 on three objects, and our construction amounts to finding the two-dimensional representation induced on a and b. Taking out the irrelevant factors we can represent u, v, and w on (x, x') (and similarly on (y, y') as follows. Define the two dimensional column vector X with the components x and x'. Then under the three discrete interchange symmetries $X \xrightarrow{u} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X$; $X \xrightarrow{v} \frac{1}{2}\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} X$; and $X \xrightarrow{w} \frac{1}{2}\begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix} X$. Imposing these transformations on the wave function (62) we obtain the conditions

$$c_{nn'} \;=\; c_{-n,n'} \tag{65}$$

$$c_{nn'} \;=\; c_{\frac{n+n'}{2},\frac{3n-n'}{2}} \tag{66}$$

and

$$c_{nn'} \;=\; c_{\frac{n-n'}{2},-\frac{3n+n'}{2}} \tag{67}$$

The conditions (63) and (65) imply that we can restrict n to range over $(0, 1, \ldots, q-1, q)$ and n' to range over $(1, 2, \ldots, 6q)$. It is convenient then to visualize a $q+1$ by $6q$ toroidal lattice (i.e. one with periodic boundary conditions) with sites labelled by (n, n') and on which a particle hops according to the rules

$$(n, n') \xrightarrow{u} (n-q, n'-3q) \tag{68}$$

$$(n, n') \xrightarrow{v} \frac{1}{2}(n+n', 3n-n') \tag{69}$$

$$(n, n') \xrightarrow{w} \frac{1}{2}(n-n', -3n-n') \tag{70}$$

We see from the rules (69) and (70) that only even lattice sites ($n \pm n' = $ even) are visited. Starting from a given site, all the sites visited by the particle by following an arbitrary sequence of u, v, and w hops, before returning to the starting site, are equivalent. We call the set of points thus visited a trip.

The desired ground state degeneracy D is equal to the number of different trips the particle can take (or equivalently, the number of inequivalent sites on the lattice.)

The reader can easily compute D pictorially for small values of q by drawing a $(q+1)$ by $6q$ lattice and hop around on it according to the hopping rules given above. For example, for $q = 3$, we have the trips

$(0,2) \sim (1,17) \sim (2,8) \sim (3,11)$,
$(0,4) \sim (2,16) \sim (3,13) \sim (1,7)$,
$(0,6) \sim (3,15)$,
$(0,8) \sim (2,14) \sim (3,17) \sim (1,5)$,
$(0,10) \sim (1,13) \sim (2,4) \sim (3,1)$,
$(0,12) \sim (3,3)$,
$(0,14) \sim (1,11) \sim (2,2) \sim (3,5)$,
$(0,16) \sim (2,10) \sim (1,1) \sim (3,7)$,
$(0,18) \sim (3,9)$,
$(1,3) \sim (1,15) \sim (2,12) \sim (1,9) \sim (2,6) \sim (2,18)$,

There are ten trips, one with six sites visited, six with four sites visited, and three with two sites visited. This inventory of the number of trips with given lengths is

also characteristic of the QH state being studied. Thus, the topological degeneracy of this QH state is $D(q = 3) = 10$. We can also check that the total number of sites visited ($= 1 \cdot 6 + 6 \cdot 4 + 3 \cdot 2 = 36$) is indeed equal to the number of even lattice sites $\frac{1}{2} 6q(q+1)|_{q=3} = 36$.

We can readily determine D for an arbitrary q. We argue that we start with a lattice whose number of sites is a quadratic function of q and that this number is reduced by various symmetry relations. So it is at least plausible that $D(q)$ is a quadratic of the form $aq^2 + bq + c$. It is simple to determine $D(q = 1) = 3$ (Laughlin's result [2] !) and $D(q = 2) = 6$, in addition to the result we showed explicitly $D(q = 3) = 10$. Fitting to these three points we find

$$D = \frac{1}{2} (q+1)(q+2) \tag{71}$$

We have verified this result by hand (for $q = 4$) and by a computer program (for a large number of q's).

We now follow the alternative of gluing the partons into electrons using an $SU(3)$ gauge field. The effective parton theory is given by

$$\mathcal{L} = i\psi^\dagger(\partial_t - ia_0)\psi + \frac{1}{2m}\psi^\dagger(\partial_i - i\frac{e}{3}A_i - ia_i)^2\psi \tag{72}$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$ and a_μ are hermitian traceless 3 by 3 matrices. Note that (56) is just the restriction of (72) to the diagonal subgroup of $SU(3)$. After integrating out the partons (with each species filling out q Landau levels), we get the $SU(3)_q$ CS theory:

$$\mathcal{L} = \frac{q}{3 \times 4\pi} A_\mu \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda} + \frac{q}{4\pi} \mathrm{Tr} \epsilon^{\mu\nu\lambda}(a_\mu \partial_\nu a_\lambda + \frac{i}{3} a_\mu a_\nu a_\lambda) \tag{73}$$

Again we choose the $a_0 = 0$ gauge. Following what we did for the $SU(2)$ case, we find the gauge invariant Wilson loops are given in terms of the following spatially constant gauge potentials (in analogy with (31)):

$$a_1(x_1, x_2, t) = 2\pi \frac{u_1(t)}{L_1}\Lambda_3 + 2\pi \frac{u_2(t)}{\sqrt{3}L_1}\Lambda_8, \quad a_2(x_1, x_2, t) = 2\pi \frac{v_1(t)}{L_1}\Lambda_3 + 2\pi \frac{v_2(t)}{\sqrt{3}L_1}\Lambda_8, \tag{74}$$

where

$$\Lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{75}$$

We see that at low energies the non-trivial physical degrees of freedom of the $SU(3)$ CS theory are described by $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ where we introduced a vector notation. After writing the effective Lagrangian for $\mathbf{u}$ and $\mathbf{v}$, we see that $\mathbf{u}$ and $\mathbf{v}$ satisfy the following commutation relation

$$[u_i, v_j] = \frac{i\delta_{ij}}{2\pi(2q)}, \quad [u_i, u_j] = 0, \quad [v_i, v_j] = 0 \tag{76}$$

The large gauge transformations

$$\begin{gathered} \exp\left(i\frac{2\pi x_1}{L_1} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \right) \\ \exp\left(i\frac{2\pi x_1}{L_1} \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right) \\ \exp\left(i\frac{2\pi x_1}{L_1} \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) \end{gathered} \tag{77}$$

and the ones in the x_2 direction lead to the following equivalence relations:

$$\mathbf{u} \sim \mathbf{u} + \mathbf{e}_i, \quad \mathbf{v} \sim \mathbf{v} + \mathbf{e}_i \tag{78}$$

where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $\mathbf{e}_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Note that the angles between the $\mathbf{e}_i$'s are 120°.

From the requirement $\psi(\mathbf{u}) \;=\; \psi(\mathbf{u} + \mathbf{e}_i)$, $i = 1, 2, 3$, we have the wave function

$$\psi(\mathbf{u}) \;=\; \sum_{\mathbf{w}} c_{\mathbf{w}}\, e^{i2\pi \mathbf{w}\cdot\mathbf{u}} \tag{79}$$

where $\mathbf{w}$ is restricted by demanding that $\mathbf{w} \cdot \mathbf{e_i}$ are integers. From the commutation relation between $\mathbf{u}$ and its conjugate momentum $\mathbf{v}$, we obtain the wave function

$$\phi(\mathbf{v}) \;=\; \sum_{\mathbf{w}} c_{\mathbf{w}}\, \delta\left(\mathbf{v} \;-\; \frac{1}{2q}\,\mathbf{w}\right) \tag{80}$$

Thus, $\phi(\mathbf{v})$ is non-vanishing only when

$$\mathbf{v} \;=\; \frac{1}{2q}\,\mathbf{w} \tag{81}$$

or upon dotting with $\mathbf{e_i}$, only when $2q\mathbf{v}\cdot\mathbf{e}_i = \text{integers}|_{i=1,2,3}$

Thus the wave function $\phi(\mathbf{v})$ is non-zero only on the dual lattice points spanned by $(\mathbf{d}_1, \mathbf{d}_2)$ which satisfy $\mathbf{d}_i\cdot\mathbf{e}_j = \delta_{ij}/2q$. Due to the periodic condition $\phi(\mathbf{v}) = \phi(\mathbf{v}+\mathbf{e}_i)$ only the circles inside the "unit cell" spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ can be independent. (See Fig. 2)

The global $SU(3)$ gauge transformations

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \to \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \to \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix} \tag{82}$$

generate a simultaneous 120° rotation on $\mathbf{u}$ and $\mathbf{v}$. This leads to the following equivalence relation:

$$(\mathbf{u},\mathbf{v}) \sim (R_{120^\circ}\mathbf{u}, R_{120^\circ}\mathbf{v}) \tag{83}$$

The global $SU(3)$ gauge transformation $(\psi_1,\psi_2,\psi_3) \to (i\psi_2, i\psi_1, \psi_3)$ also generate a transformation $(u_1,u_2,v_1,v_2) \to (-u_1,u_2,-v_1,v_2)$ and the corresponding equivalence relation.

All the above equivalence relations can be satisfied by requiring the wave function to satisfy

$$\psi(\mathbf{v}) = \psi(R_{120^\circ}\mathbf{v}), \quad \psi(v_1,v_2) = \psi(-v_1,v_2) \tag{84}$$

Now only the black circles in the "unit cell" are independent (see Fig. 2). The number of the ground states (the black circles) can be calculated as follows. First the number of the black circles inside the shaded triangle (see Fig. 2) is $\frac{q^2}{2}$. However each of the $3(q-1)$ black circles on the edge is only counted as $1/2$ in the above calculation. To include the other half we need to add $3\frac{q-1}{2}$. The 3 black circles on the corners is only counted as $1/6$ each. So we also need to include a term $3\frac{5}{6}$. Thus the total number of states is $\frac{q^2}{2} + 3\frac{q-1}{2} + 3\frac{5}{6} = \frac{(q+1)(q+2)}{2}$ in agreement with (71).

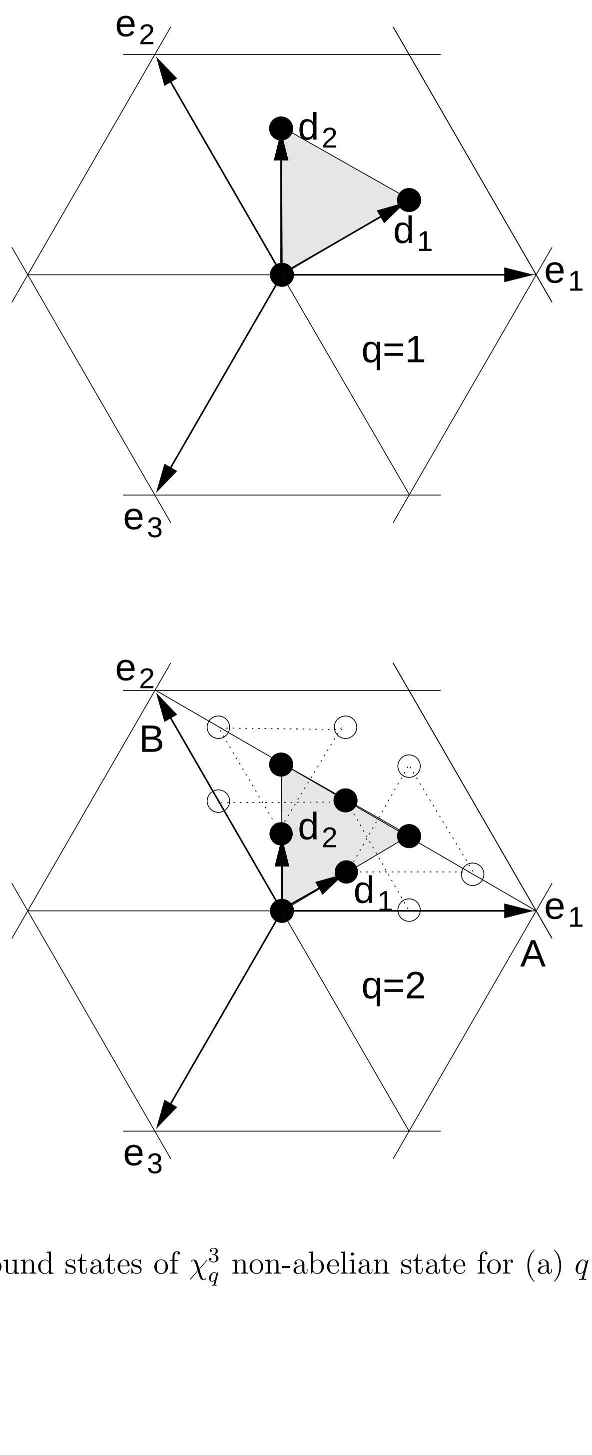

Figure 2: The ground states of χ_q^3 non-abelian state for (a) $q = 1$ and (b) $q = 2$.

6 A general picture

A general picture emerges from these calculations. We can now propose the following complete classification scheme for the ground state structures for a large class of QH states on a torus, abelian or non-abelian: For all QH states reachable by the parton construction, the ground states on the torus are characterized by a lattice and the hopping rules on this lattice.

To be more precise, the ground states are labeled by lattice points in a vector space. The equivalent lattice points label the same physical ground state. There are two kinds of equivalence relations: A) Translation:

$$\mathbf{v} \sim \mathbf{v} + \mathbf{e}_i|_{i=1,\dots,\mathrm{Dim}(\mathbf{v})} \tag{85}$$

and B) linear map:

$$\mathbf{v} \sim M_i \mathbf{v}|_{i=1,2,\dots} \tag{86}$$

where M_i is a $\mathrm{Dim}(\mathbf{v})$ by $\mathrm{Dim}(\mathbf{v})$ matrix.

$\mathbf{v}$ have conjugate variables, $\mathbf{u}$, which satisfy the same equivalence relations (85) and (86). The commutator between $\mathbf{v}$ and $\mathbf{u}$ has the form

$$[v_i, u_j] = i g_{ij}/2\pi \tag{87}$$

The symmetric matrix $g = (g_{ij})$ defines an inner product $\mathbf{v}_1 \cdot \mathbf{v}_2 \equiv v_{1_i}(g^{-1})_{ij} v_{2_j}$ (which may not be positive definite). The lattice that labels the ground states is just the dual lattice (which will be called the $\mathbf{d}$-lattice) of the lattice generated by basis vectors $\mathbf{e}_i$ (which will be called the $\mathbf{e}$-lattice). The basis vectors $\mathbf{d}_i$ of the $\mathbf{d}$-lattice are given by

$$\mathbf{d}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{88}$$

For this to be consistent with the equivalence relation (85), we also require that the $\mathbf{e}$-lattice is a sub-lattice of the $\mathbf{d}$-lattice. In other words, $K_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ must be an integer for all i and j. This also implies that

$$\mathbf{d}_i \cdot \mathbf{d}_j = (K^{-1})_{ij}, \tag{89}$$

where K is matrix: $K = (K_{ij})$. We would like to remark that different choices of basis for $\mathbf{v}$ and $\mathbf{u}$ lead to different g_{ij}. But K_{ij} does not depend on the choice of the basis. In our discussion of the $SU(3)$ QH liquid, g_{ij} was taken to be $\delta_{ij}/2q$ (see (76)).

For abelian states, only the equivalence relation (85) appears and the K-matrix completely describes the ground state structures. The ground states are labeled by the points on the $\mathbf{d}$-lattice inside the unit cell of the $\mathbf{e}$-lattice. However, for non-abelian states, we also have an additional type of equivalence relation of the form in (86). This completely changes the structure of the ground states. Only part of the points in the unit cell correspond to independent ground states.

We see that the ground state structure of a QH state on a torus is described by a lattice characterized by the K-matrix (89) plus a set of linear maps within the lattice.

Now let us discuss what kind of linear maps are allowed. For convenience, we choose the basis for $\mathbf{v}$ and $\mathbf{u}$ such that $g_{ij} = (K^{-1})_{ij}$. This is always possible since g_{ij} and K_{ij} have the same signature (*ie* the same number of positive eigenvalues and the same number of negative eigenvalues). With this choice of basis, $\mathbf{e}_i$ become the standard basis vector: the j^{th} elements of $\mathbf{e}_i$ is just $(\mathbf{e}_i)_j = \delta_{ij}$.

To obtain the condition on the maps M_i, first we note that the map M_i acts on both $\mathbf{v}$ and $\mathbf{u}$ and keep the commutator (87) unchanged. Thus M_i must leave g_{ij} or in our case K_{ij} invariant:

$$M_i^T K M_i = K \tag{90}$$

This implies that $\det(M_i) = \pm 1$. M_i should also map the $\mathbf{e}$-lattice onto itself. This requires M_i to be an integer matrix. Thus the allowed maps M_i are elements in $L(\mathrm{Dim}(K), Z)$ which leaves K invariant (90). To obtain the M_i's, we may start with a transformation between the partons which leave the electron operator unchanged, as we did for the $U(1)_{2q+4} \times SU(2)_q/Z_2$ state. Such a transformation induces a transformation on the gauge fields, and hence a transformation on $\mathbf{v}$'s and $\mathbf{u}$'s which is nothing but the M_i transformation.

It is interesting to work out the K matrix for the $U(1)_{2q+4} \times SU(2)_q/Z_2$ and the $SU(3)_q$ states. For the $U(1)_{2q+4} \times SU(2)_q/Z_2$ state we have $g = \begin{pmatrix} 1/(2q+4) & 0 \\ 0 & 1/2q \end{pmatrix}$,

$\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (1/2, -1/2)$ (see (43), (44), and (47)). Thus $K = \begin{pmatrix} 2q+4 & q+2 \\ q+2 & 1 \end{pmatrix}$. For the $SU(3)_q$ state, we have $g = \begin{pmatrix} 1/2q & 0 \\ 0 & 1/2q \end{pmatrix}$, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (-1/2, \sqrt{3}/2)$. Thus $K = \begin{pmatrix} 2q & -q \\ -q & 2q \end{pmatrix}$.

In our previous work [6] we have emphasized that, for abelian states, the filling fraction ν, even when supplemented by the topological degeneracy D, cannot (evidently) capture all the information contained in the matrix K. Similarly for the non-abelian states. For example, we note that the $U(1)_{2q+4} \times SU(2)_q/Z_2$ state studied in section 4 has degeneracy $D = \frac{1}{2}(q+1)(q+2)$, exactly the same as the degeneracy in (71) for the $SU(3)_q$ states. However the two states are in general different. For one thing the $U(1)_{2q+4} \times SU(2)_q/Z_2$ state has a filling fraction $\nu = \frac{q}{q+2}$, while the $SU(3)_q$ state has $\nu = \frac{q}{3}$. But, for $q = 1$, we have the same ν and D, and indeed they both correspond to the Laughlin $\nu = \frac{1}{3}$ state even though their effective theories are quite different: $SU(3)_1$ has eight gauge potentials, while $U(1)_6 \times SU(2)_1$ has only four.

We trust that the reader can now work out the lattice and the hopping rules for any non-abelian states reachable by the parton construction. For a state described by the group G, the dimensions of the lattice is given by the rank of G. For example, if we cut the electron into five equal pieces and construct the $SU(5)_q$ states, we would have a particle hopping on a four dimensional lattice.

7 Summary

In this paper we propose a simple method to calculate the ground state degeneracy of several non-abelian QH liquids. Our method can be applied to any abelian and non-abelian QH liquids obtained from the parton construction. A general pattern emerge from our calculation. For the QH liquids reachable by the parton construction, the ground states on a torus can be described by points on a lattice. For abelian QH liquids, the ground states correspond to points inside a "unit cell". For non-abelian states the ground states correspond to points inside a "folded" unit cell. The folding

is generated by reflection, and possibly rotations. (See Fig. 1 and Fig. 2)

8 Acknowledgements

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9 Appendix: Coset Construction

In this Appendix, we are going to study the coset construction of the CS theory. We will see that the coset construction is closely related to the parton construction.

We start with a physical example. The $\nu = 1/k$ Laughlin state can be constructed from a parton construction [13]. For example, to construct the $\nu = 1/2$ Laughlin state (of bosons), we may start with two kinds of partons in the $\nu = 1$ state, with the wave function $\Psi \sim \prod_{ij}(z_i^{(1)} - z_j^{(1)}) \prod_{ij}(z_i^{(2)} - z_j^{(2)})$ where $z_i^{(1)}$ and $z_i^{(2)}$ represent the coordinates of the two kinds of partons, described by a CS effective theory (1) with $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This CS effective theory is thus the $U(1)_1 \times U(1)_1$ CS theory.

Suppose we now make a projection by setting $z_i^{(1)} = z_i^{(2)}$. Physically, we recombine the two kinds of partons together into electrons. The wave function becomes $\Psi \sim \prod_{ij}(z_i - z_j)^2$ and the parton state becomes the $\nu = 1/2$ Laughlin state.

We will now describe how this projection can be achieved in the $U(1)_1 \times U(1)_1$ CS effective theory. Note that the projection binds the two kinds of partons so that they always move together. Thus after projection, their currents and densities $j_\mu^{(1)} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^{(1)}$ and $j_\mu^{(2)} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^{(2)}$ must be equal. This leads to the constraint $a_\mu^{(1)} - a_\mu^{(2)} = 0$. By setting $a_\mu^{(1)} = a_\mu^{(2)}$ in $\mathcal{L} = \frac{1}{4\pi}(\epsilon^{\mu\nu\lambda} a_\mu^{(1)} \partial_\nu a_\lambda^{(1)} + \epsilon^{\mu\nu\lambda} a_\mu^{(2)} \partial_\nu a_\lambda^{(2)})$ we see that the $U(1)_1 \times U(1)_1$ CS theory is reduced to the $U(1)_2$ CS theory which is the effective theory of the $\nu = 1/2$ Laughlin state.

Next we would like to understand how the parton-construction realizes itself in the ground states of the QH liquids. In other words, we would like to start with the ground states of the $U(1)_1 \times U(1)_1$ CS theory, and to see how to "project" to obtain

the ground states of the $\nu = 1/2$ Laughlin state (*i.e.* the ground states of the $U(1)_2$ CS theory). It turns out that this can be achieved through the coset construction.

We know that before projection, the edge excitations of the parton state (described by the $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ CS theory) is described by two currents $j_{1,2}$ associated with each kind of partons. The currents form two level-1 $U(1)$ Kac-Moody (KM) algebras: $U(1)_1 \times U(1)_1$. The relative motion of the two kinds of partons is described by $j_- = j_1 - j_2$ which forms a level-2 $U(1)$ KM algebra: $U(1)_2$. (Schematically, $[j_1, j_1] \sim 1$ and $[j_2, j_2] \sim 1$ and hence $[j_1 - j_2, j_1 - j_2] \sim 2$). After projecting out the the $U(1)_2$ KM algebra, the resulting edge theory is described by the coset model $U(1)_1 \times U(1)_1 / U(1)_2$ with a $U(1)_2$ KM algebra generated by $j_+ = j_1 + j_2$.

Motivated by the relation between the edge theory and the CS theory, we would like to study the coset theory of the CS theory. In particular, we would like to show that the coset CS theory $U(1)_1 \times U(1)_1 / U(1)_2$ is nothing but the $U(1)_2$ CS theory, as suggested by the parton construction given above.

Let us first start with a simpler problem. Consider a $U(1)_k$ CS theory, which has k degenerate ground states on the torus. The coset $U(1)_k / U(1)_k$ model can be obtained by making the projection $a_\mu = 0$. Obviously, after projection there is no non-trivial low energy excitation and the ground state is non-degenerate.

The ground states of the $U(1)_k$ CS theory are described by x and y operators satisfying the commutator (9). First we note that x and y operators do not satisfy the equivalence condition (6), and hence are not operators that act within the the physical Hilbert space. The allowed operators are $U_{mn} \equiv e^{2\pi i(mx+ny)}$ with $m, n =$integers. Naively one may want to project by requiring the states to satisfy $x\psi = y\psi = 0$. Or more precisely, one must require $U_{10}\psi = U_{01}\psi = \psi$. Since x and y (or U_{10} and U_{01}) do not commute (with a commutator equal to a number), no state satisfies this condition. Thus no state survives the projection. This contradicts our physical intuition that there should be one and only one state that survives the projection.

To construct the projected theory $U(1)_k / U(1)_k$ correctly, we start with a $U(1)_k \times$

$U(1)_{-k}$ theory described by the effective Lagrangian

$$L \;=\; 2\pi \; k \; (\dot{x} \; y \;-\; \dot{x}' \; y') \tag{91}$$

describing two particles living on the torus. The $U(1)_k$ in $U(1)_k \;\times\; U(1)_{-k}$ is the theory before projection and $U(1)_{-k}$ is the "conjugate" of the projection and hence a $U(1)_{-k}$ theory. (In general, to construct the G/H coset model, one may start with the $G \times H^*$ model.) We have

$$[x, y] \;\;=\;\; \frac{i}{2\pi k} \tag{92}$$

and

$$[x', y'] \;\;=\;\; -\frac{i}{2\pi k} \tag{93}$$

thus implying

$$[x - x', \; y - y'] \;\;=\;\; 0 \tag{94}$$

As in (12) and (14), the wave functions are given by

$$\psi(y, y') \;\;=\;\; \sum_{n,n'} c_{nn'} \; e^{i \; 2\pi(ny \;+\; n'y')} \tag{95}$$

and

$$\phi(x, x') \;\;=\;\; \sum_{n,n'} c_{nn'} \; \delta(kx \;-\; n) \; \delta(-kx' \;-\; n') \tag{96}$$

Periodicity $x \sim x + 1$ and $x' \sim x' \;+\; 1$ imply, as in (15),

$$c_{nn'} \;\;=\;\; c_{n+k, \; n'} \;\;=\;\; c_{n,n' \;+\; k} \tag{97}$$

This leads to a total of $k \times k$ ground states before the projection.

We now project. Remember that we want to impose $x \;=\; y \;=\; 0$. But this is impossible since x and y do not commute. With the help of the additional sector $U(1)_{-k}$, we can impose the conditions

$$x \;-\; x' \;=\; 0 \tag{98}$$

and

$$y \;-\; y' \;=\; 0 \tag{99}$$

This is possible since $x-x'$ and $y-y'$ commute. In the language of (91) the projection has the physical interpretation of binding the two particles together. Since the coordinates on the torus are defined only mod integer, the right hand sides of (98) and (99) should be interpreted as 0 mod integer. (Or more precisely, we can only impose the condition on the allowed operators: $e^{i2\pi(x-x')} = e^{i2\pi(y-y')} = 1$.) Thus the states in the projected theory satisfy

$$e^{i2\pi(x-x')}\psi = e^{i2\pi(y-y')}\psi = \psi \tag{100}$$

Writing (96) as

$$\psi(x,x') \;=\; \sum_{n,n'} c_{n,n'}\;\delta\left(k(x-x') \;-\; (n+n')\right)\;\delta(kx' \;+\; n') \tag{101}$$

we see that the $c_{nn'}$'s are non-zero only when

$$n \;+\; n' \;=\; 0 \; mod \; k. \tag{102}$$

Next, writing (95) as

$$\psi(y,y') \;=\; \sum_{n,n'} c_{n,n'}\; e^{i\;2\pi((n\;+\;n')\;y\;+\;n'\;(y'\;-\;y))} \tag{103}$$

we see, referring to (16), that in order for $\psi(y,y')$ to be proportional to $\delta^P(y-y')$ as required by (99), the $c_{nn'}$'s should depend only on $n \;+\; n'$:

$$c_{nn'} \;=\; d_{n+n'} \tag{104}$$

Referring to (102) and to the periodicity condition (97) we see that $d_{n+n'}$ is independent of $n+n'$. All the non-zero $c_{nn'}$'s are equal. There is only one state left after projection, as expected for the $U(1)_k/U(1)_k$ coset model. Thus, in contrast to the naive construction described before, we have now managed to obtain the correct result using this construction starting with the $U(1)_k \;\times\; U(1)_{-k}$ theory.

Using the fact that $n+n' \;=\; jk$ has to be a multiple of k, we can now obtain

$$\begin{aligned}\psi(y,y') \;&=\; \sum_j \; e^{i\;2\pi\;j\;k\;y} \sum_{n'} \; e^{i\;2\pi n'\;(y-y')} \\ &=\; \delta^P\;(ky)\;\delta^P\;(y-y')\end{aligned} \tag{105}$$

Similarly, we can write (101) as

$$\begin{aligned}\phi(x,x') &= \sum_j \delta\left(k(x-x') \;-\; kj\right) \sum_{n'} \delta(kx' \;+\; n') \\ &= \delta^P(kx')\,\delta^P(x-x') \qquad (106)\end{aligned}$$

We see explicitly that $\phi(x,x')$ and $\psi(y,y')$ have the same status. Indeed, note

$$\delta^P(kx)\,\delta^P(x-x') \;=\; \delta^P(kx')\,\delta^P(x-x') \qquad (107)$$

We remark in passing the obvious fact that $\delta^P(kx)\delta^P(x-x')$ is not equal to $\delta^P(kx)\,\delta^P(kx')$

The above example suggests that to construct a G/H coset model, one may start with an enlarged theory $G\times H^*$ and then project out a diagonal part $H\times H^*$ to end up with G/H. Such an approach at least works for the coset construction of KM algebras.

Now we are ready to consider the $U(1)_1\times U(1)_1/U(1)_2$ coset model. Let us study the more general $U(1)_k\times U(1)_{k'}/U(1)_{k+k'}$ coset model where k and k' have no common factors. Starting from the $U(1)_k\times U(1)_{k'}\times U(1)_{-(k+k')}$ CS theory described by the operators (x,y), (x',y'), and (x'',y''):

$$[x,y]=\frac{i}{2\pi k},\quad [x',y']=\frac{i}{2\pi k'},\quad [x'',y'']=-\frac{i}{2\pi(k+k')}, \qquad (108)$$

we project by imposing the conditions

$$kx \;+\; k'x' \;+\; k''x'' =\; 0 \qquad (109)$$

and

$$ky \;+\; k'y' \;+\; k''y'' =\; 0 \qquad (110)$$

where we have defined for notational convenience $k''\equiv-(k+k')$. This is possible since $kx\;+\;k'x'\;+\;k''x''$ and $ky\;+\;k'y'\;+\;k''y''$ commute. Again the right hand sides of (109) and (110) should be interpreted as 0 mod integer, because the coordinates (x,y) etc. are only defined up to an integer.

Extending (95) and (96), we write the wave functions as

$$\psi(y,y',y'') \;=\; \sum_{n,n',n'} c_{nn'n''}\; e^{i\,2\pi(ny\;+\;n'y'\;+\;n''y'')} \qquad (111)$$

and

$$\phi(x,x',x'') \;=\; \sum_{n,n',n''} c_{nn'n''}\; \delta(kx \;-\; n)\; \delta(k'x' \;-\; n')\; \delta(k''x'' \;-\; n'') \tag{112}$$

Periodicity $x \sim x+1$ etc. imply, as in (97),

$$c_{nn'n''} \;=\; c_{n+k,\; n'n''} \;=\; c_{n,n'\;+\;k,n''} \;=\; c_{n,n',n''+k''} \tag{113}$$

This leads to a total of $kk'k''$ states before projection.

It turns out that the projection does not remove any states. Indeed, we see from (112) that (109) is already satisfied, posing no restriction on the $c_{nn'n''}$'s. Defining $m = [n]_k$ as before and the corresponding primed and double primed quantities m' and m'' we write (Cf (18)

$$\psi(y,y',y'') \;=\; \left(\sum_{m,m',m''=1}^{k,k',k''} c_{m,m',m''}\; e^{i2\pi(my+m'y'+m''y'')}\right)\; \delta^P(ky)\delta^P(k'y')\delta^P(k''y'') \tag{114}$$

We find that (110) does not impose any further restriction either. Thus the $U(1)_k \times U(1)_{k'}/U(1)_{k+k'}$ coset model has $kk'(k+k'')$ degenerate ground states. In particular, the $U(1)_1 \times U(1)_k/U(1)_{k+1}$ coset model has $k(k+1)$ degenerate ground states, in agreement with a previous result obtained using a more abstract and algebraic approach[10]. When $k=1$ the $U(1)_1 \times U(1)_1/U(1)_2$ coset model has 2 degenerate ground states which is the same as the number of ground states in the $U(1)_2$ theory and the $\nu = 1/2$ Laughlin state.

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